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On the Robustness of the Damped V-Cycle of the Wavelet Frequency Decomposition Multigrid Method

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Abstract — Zusammenfassung

On the Robustness of the Damped V-Cycle of the Wavelet Frequency Decomposition Multigrid Method. The damped V-cycle of the wavelet variation of the "Frequency decomposition multigrid method" of Hackbusch [Numer. Math. *56*, pp. 229–245 (1989)] is considered. It is shown that the convergence speed under sufficient damping is not affected by the presence of anisotropy but still depends on the number of levels. Our analysis is based on properties of wavelet packets which are supplied and proved. Numerical approximations to the speed of convergence illustrate the theoretical results.

AMS Subject Classifications: 65F10, 65N30

Key words: Wavelets, wavelet packets, robust multilevel methods, V-cycle.

Zur Robustheit des gedämpften V-Zyklus bei der FDMGM mit Wavelets. Wir betrachten den gedämpften V-Zyklus für die Wavelet-Variante der "Frequenzzerlegungs-Multigridmethode" von Hackbusch [Numer. Math. *56*, 229–245 (1989)]. Es wird gezeigt, daß die Konvergenzgeschwindigkeit bei hinreichender Dämpfung durch Anisotropie nicht beeinflußt wird, aber noch von der Anzahl des Niveaus abhängt. Unsere Analyse beruht auf Eigenschaften von Wavelet-Paketen, die formuliert und bewiesen werden. Numerische Schätzungen der Konvergenzgeschwindigkeit erläutern die theoretischen Ergebnisse.

1. Introduction

An iterative method for solving a linear system arising by the discretization of the *anisotropic* model problem

$$-\varepsilon\frac{\partial^2}{\partial x^2}u(x,y)-\frac{\partial^2}{\partial y^2}u(x,y)+u(x,y)=f(x,y)\quad\text{in}\quad \Omega=(0,s)^2, \tag{1.1}$$

$$u \quad \text{periodic}, \tag{1.2}$$

$0<\varepsilon\le 1$, is said to be *robust* if its rate of convergence (i.e. spectral radius of the iteration matrix) is bounded smaller than 1 uniformly in ε and in the discretization step-size.

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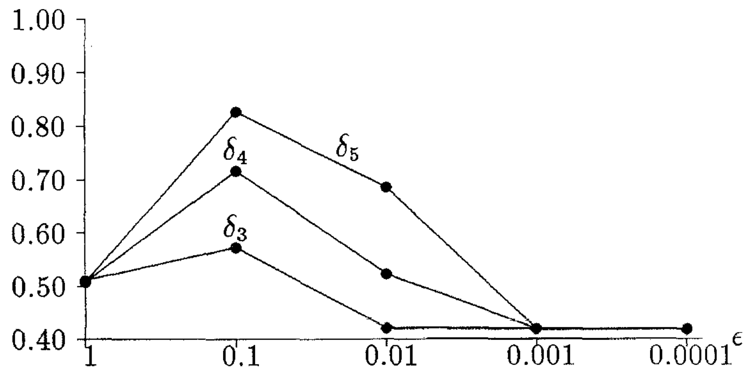

Figure 1. V-cycle convergence rates of the wavelet variation of the FDMGM for different discretization step-sizes: $\delta_l = 2^{-l}$

A wavelet variation of the frequency decomposition mutigrid method (FDMGM) of Hackbusch [11] was presented in [16] and the robustness of the corresponding two-level method with respect to any intermediate level was verified.

However, in numerical experiments, the V-cycle variant of the multilevel method appeared not to be robust, see Fig. 1 (for a more detailed description of the experiment presented in Fig. 1 see Section 3.1). In this paper we are able to prove that a sufficiently damped version of the V-cycle is *almost robust* in the sense that the rate of convergence depends at most on the number of levels. Our result is of the same quality as typical estimates known for the convergence rates of multigrid solvers without regularity assumptions on the continuous problem, see [1], [3] and [17]. Indeed, the situation without regularity assumptions and the situation for the FDMGM are comparable in their lack of the *approximation property*. Hence, the standard proofs for the V-cycle convergence, see e.g. [10], are not applicable and adequate modifications are required which lead to level dependent convergence rates.

In the following section we shortly introduce the necessary vocabulary (wavelet packets, Mallat transformation, wavelet-Galerkin discretization) to understand the considerations in Section 3. Here, we prove the convergence of the damped V-cycle. Basically, we use the same techniques as in Chapter 7.2 of [10]. For the ease of presentation two auxillary results which are crucial but rather technical are given in the Appendix.

2. Wavelet Analysis

2.1. *Wavelet Packet System*

We will briefly recall various definitions and properties of wavelet packets [5] generated by the Daubechies scaling functions constructed in [6]. For a positive integer N, the Daubechies wavelet packets $\{\psi^l\}_{l\in\mathbf{N}_0}$ of order N are defined as follows: There exist $2N$ real numbers a_i, $i = 0, 1, \ldots, 2N-1$, satisfying

$$\sum_k a_k = 2 \quad \text{and} \quad \sum_k a_{k+2m}a_k = 2\delta_{0,m} \quad \text{for all } m \in \mathbf{Z}, \tag{2.1}$$

so that

$$\psi^{2l}(x) = \sum_{k=0}^{2N-1} a_k \psi^l(2x-k), \tag{2.2}$$

$$l = 0, 1, 2, \ldots$$

$$\psi^{2l+1}(x) = \sum_{k=0}^{2N-1} b_k \psi^l(2x-k), \tag{2.3}$$

where $b_k = (-1)^k a_{2N-k-1}$. The function ψ^0 is called *scaling function* and ψ^1 is called *wavelet*. All wavelet packets are compactly supported, with $\mathrm{supp}(\psi^l) = [0, 2N-1]$. For convenience, we define $a_k = 0$ for $k \notin [0, 2N-1]$.

The wavelet packets are in $C^{\alpha(N)}$, the space of Hölder continuous functions with exponent $\alpha(N)$, where $\alpha(2) \approx .55$, $\alpha(3) \approx 1.09$ and $\alpha(N) \approx 0.2075 \cdot N$ for large N [9]. We refer to [15] for comprehensive introduction to wavelet packets.

For the multilevel process defined in Section 3 we will need some of the second order *connection coefficients* ([2], [13])

$$\Gamma_k^l := \int_{\mathbf{R}} (\psi^l)'(x-k)(\psi^l)'dx, \quad 2-2N \le k \le 2N-2, \quad l \ge 0, \tag{2.4}$$

of wavelet packets with $N \ge 3$. Due to the recursive definition of wavelet packets, their conection coefficients can be easily computed from the connection coefficients Γ_k^0 of the scaling function ψ^0.

2.2. *Mallat Transformations*

The (periodic) *Mallat transformations* $h, g\colon \mathbf{R}^n \to \mathbf{R}^{n/2}$, n even, of a vector $v \in \mathbf{R}^n$ are defined by

$$(hv)_k = \frac{1}{\sqrt{2}} \sum_{l=0}^{2N-1} a_l v_{l+2k}, \tag{2.5}$$

$$k = 0, 1, \ldots, n/2 - 1,$$

$$(gv)_k = \frac{1}{\sqrt{2}} \sum_{l=0}^{2N-1} b_l v_{l+2k}, \tag{2.6}$$

where we extend v periodically, i.e. $v_l = v_{l+n}$. The coefficients a_l in (2.5) and b_l in (2.6) are those in (2.2) and (2.3), respectively. The Mallat transformations satisfy (see [6], [14]),

$$h^t h + g^t g = I\,,$$
$$h h^t = g g^t = I\,,$$
$$g h^t = h g^t = 0\,.$$

We use I to denote the identity matrix of appropriate size throughout this paper.

2.3. Wavelet-Galerkin Discretization of the Model Problem

We introduce the Sobolev space $H_p^1(\Omega)$, $\Omega = (0, s)^2$, with periodic boundary conditions,

$$H_p^1 = H_p^1(\Omega) := \{v \in L^2(\Omega)\colon v_x, v_y \in L^2(\Omega), v(0, y) = v(s, y), v(x, 0) = v(x, s)\}\,.$$

The weak or variational formulation of the model problem (1.1), (1.2), reads:

$$\text{find } u \in H_p^1\colon \mathscr{A}(u, v) = \int_\Omega f v\, dx dy, \quad \text{for all } v \in H_p^1\,, \tag{2.7}$$

where $\mathscr{A}$ is the H_p^1-elliptic bilinear form

$$\mathscr{A}(u, v) = \int_\Omega (\varepsilon u_x v_x + u_y v_y + uv)\, dx dy\,.$$

Due to the Lax-Milgram theorem [4] (2.7) has a unique solution u. For a wavelet-Galerkin discretization of (2.7), we assume that $N \geq 3$ and that s in (1.1) is an integer greater than $4N - 3$. Further, we set $\psi_{l,k}^0(x) = 2^{l/2}\psi^0(2^l x - k)$ and introduce the wavelet-Galerkin spaces

$$V_l = V_l(0, s) := \left\{v \in L^2(0, s)\colon v(x) = \sum_{k \in \mathbf{Z}} c_k \psi_{l,k}^0(x),\ x \in [0, s], \text{ and } c_k = c_{k+2^l s}\right\}.$$

Obviously, V_l has the dimension $n_l = 2^l s$. The wavelet-Galerkin approximation u_l to u in the tensor product space $V_l \otimes V_l \subset H_p^1$ is the unique solution of

$$\mathscr{A}(u_l, v_l) = \int_\Omega f v_l\, dx dy, \quad \text{for all } v_l \in V_l \otimes V_l\,.$$

A convergence proof of $u_l \to u$ is given in [18]. For u_l we have the following expansion $u_l(x, y) = \sum_{i,j} u_{i,j}^l \psi_{l,i}^0(x)\psi_{l,j}^0(y)$ where the expansion coefficients $u_{i,j}^l$ are periodic with period n_l in each index. We now define $f_{i,j}^l = \int_\Omega f(x, y)\psi_{l,i}^0(x)\psi_{l,j}^0(y)\, dx dy$. If we order the $u_{i,j}^l$'s and $f_{i,j}^l$'s, $0 \leq i, j \leq n_l - 1$, lexicographically and denote the resulting vectors U_l and F_l, respectively, then we have the following linear system for the n_l^2 unknowns U_l,

$$A_{l,0} U_l = F_l\,, \tag{2.8}$$

where ($\otimes$ denotes the tensor product of spaces, operators and vectors)

$$A_{l,0} = \varepsilon c_l^0 \otimes I + I \otimes c_l^0 + I \otimes I \tag{2.9}$$

is the Galerkin approximation (stiffness matrix) of $\mathscr{A}$ in $V_l \otimes V_l$. In (2.9) c_l^0 is a symmetric and circulant $n_l \times n_l$ matrix [8], which is completely determined by the connection coefficients Γ_k^0 (2.4),

$$c_l^0 = \delta_l^{-2} \begin{bmatrix} \Gamma_0^0 & \Gamma_1^0 & \cdots & \Gamma_p^0 & 0 & 0 & \cdots & 0 & \Gamma_p^0 & \cdots & \Gamma_2^0 & \Gamma_1^0 \\ \Gamma_1^0 & \Gamma_0^0 & \cdots & \cdots & \Gamma_p^0 & 0 & \cdots & 0 & 0 & \Gamma_p^0 & \cdots & \Gamma_2^0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Gamma_1^0 & \cdots & \Gamma_p^0 & 0 & 0 & 0 & \cdots & \Gamma_p^0 & \cdots & \cdots & \Gamma_1^0 & \Gamma_0^0 \end{bmatrix},$$

where $p = 2N - 2$ and $\delta_l = 2^{-l}$ is the *discretization step-size.* We denote a circulant matrix by its first row,

$$c_h^0 = \delta_l^{-2} \operatorname{Cir}_{n_l}(\Gamma_0^0 \quad \Gamma_1^0 \quad \cdots \quad \Gamma_p^0 \quad 0 \quad \cdots \quad 0 \quad \Gamma_p^0 \quad \cdots \quad \Gamma_1^0).$$

3. Multilevel Scheme

3.1 Definitions and Notation

For the definition of the FDMGM we supply some notation. A more detailed discussion of the wavelet variation of the FDMGM can be found in [16], see also the original paper [11]. First, we define the matrices $A_{l,m} \in \mathbf{R}^{n_l^2 \times n_l^2}$ by

$$A_{l,m} := \varepsilon c_l^m \otimes I + I \otimes c_l^0 + I \otimes I, \tag{3.1}$$

where

$$c_l^m = \delta_l^{-2} \operatorname{Cir}_{n_l}(\Gamma_0^m \quad \cdots \quad \Gamma_{2N-2}^m \quad 0 \quad \cdots \quad 0 \quad \Gamma_{2N-2}^m \quad \cdots \quad \Gamma_1^m) \tag{3.2}$$

with the connection coefficients Γ_k^m (2.4) of the wavelet packet ψ^m (2.2) resp. (2.3). We have that

$$A_{l-1,2m} = (h \otimes h) A_{l,m} (h^t \otimes h^t) \quad \text{and} \quad A_{l-1,2m+1} = (g \otimes h) A_{l,m} (g^t \otimes h^t),$$

where h and g are the Mallat transformations (2.5) and (2.6) of appropriate dimension. All of the matrices $A_{l,m}$ are positive definite.

The iteration matrix of the *basic iterative method* (BIM) (also called *smoothing iteration* [10]) with respect to $A_{l,m}$ will be denoted by

$$S_{l,m} = I - W_{l,m}^{-1} A_{l,m}. \tag{3.3}$$

The matrix $W_{l,m}$ characterizes the special BIM, for instance the choice $W_{l,m} = \vartheta_{l,m}^{-1} \operatorname{diag}(A_{l,m})$, $\vartheta_{l,m} \in \mathbf{R}$, gives the daped Jacobi iteration.

Now, we are able to formulate the recursive multilevel procedure. We let $L_e \geq 0$ be the coarsest level, that is the level where the corresponding linear systems are solved exactly, and we let L be the level number with respect to the finest discretization. We have four variable quantities in our procedure:

- l is the varying level, $L_e \leq l \leq L$,
- m is a non-negative integer on level l, $0 \leq m \leq 2^{L-l} - 1$,
- $w \in \mathbf{R}^{n_l^2}$ is a starting guess,
- $b \in \mathbf{R}^{n_l^2}$ is a right hand side or a defect.

We then define the following recursive multilevel procedure MLP:

$$
\begin{aligned}
&\mathbf{MLP}(l, m, w, b)\\
&\mathbf{begin}\\
&\mathbf{if}\ l = L_e\ \mathbf{then}\ w := A_{l,m}^{-1} b\ \mathbf{else}\\
&w := S_{l,m}^{\nu_1} w + \sum_{j=0}^{\nu_1 - 1} S_{l,m}^{j} W_{l,m}^{-1} b \quad (\nu_1 \text{ steps of the BIM})\\
&d := A_{l,m} w - b,\ v_0 = 0,\ v_1 = 0\\
&\mathbf{MLP}(l-1, 2m, v_0, (h \otimes h)d)\\
&\mathbf{MLP}(l-1, 2m+1, v_1, (g \otimes h)d)\\
&w := w - \beta_m (h^t \otimes h^t) v_0 - \beta_m (g^t \otimes h^t) v_1 \qquad (3.4)\\
&w := S_{l,m}^{\nu_2} w + \sum_{j=0}^{\nu_2 - 1} S_{l,m}^{j} W_{l,m}^{-1} b \quad (\nu_2 \text{ steps of the BIM})\\
&\mathbf{end}
\end{aligned}
$$

The above procedure describes a damped V-cycle with *damping parameters* $\beta_m \geq \mathbf{R}$. One step of the multilevel method for solving (2.8) on level $L \geq 1$ is performed by

$$
\begin{aligned}
w &:= U_L^{\mu},\\
&\mathrm{MLP}(L, 0, w, F_L), \qquad (3.5)\\
U_L^{\mu+1} &:= w.
\end{aligned}
$$

The V-cycle is an $O(N \cdot n_L^2)$-algorithm, see [11] and [16]. It is assumed that the connection coefficients are precomputed and stored. The precomputation of the connection coefficients $\{\Gamma_k^m\}_{m=0,\ldots,2^L}$ requires the solution of a linear system of dimension $4N - 3$ for the Γ_μ^0's [13]; the other connection coefficients are generated from these by Mallat transformations involving $O(2^L \cdot N^2)$ operations.

The convergence rates of the iteration (3.5) without damping ($\beta_m \equiv 1$) for $L \in \{3, 4, 5\}$ with $L_e = 0$ and for the Daubechies wavelet packets $N = 3$ are plotted in Fig. 1. We used the damped Jacobi iteration as BIM with damping parameters $\vartheta_{l,m} = \Gamma_0^m / \sum_k |\Gamma_k^m|$, see [16] for an explanation, and with $\nu_1 = 2$ and $\nu_2 = 0$. The Gauß-Seidel relaxation as BIM gives the same qualitative behaviour of the convergence speed.

Next, we consider the iteration matrix $M_{L,0}(\nu_1, \nu_2)$ of the iteration (3.5). Let $M_{l,m}(\nu_1, \nu_2)$ denote the iteration matrix of the iteration

$$
\begin{aligned}
w &:= U_l^{\mu},\\
&\mathrm{MLP}(l, m, w, F_l), \qquad (3.6)\\
U_l^{\mu+1} &:= w,
\end{aligned}
$$

that is, we start our multigrid procedure to solve $A_{l,m} U_l = F_l$ on level l.

Lemma 3.1. *The above defined matrices $M_{l,m}(\nu_1, \nu_2)$, $L_e < l \leq L$, for the iteration (3.6) satisfy the recursion*

$$\begin{aligned} &M_{l,m}(\nu_1, \nu_2) \\ &\quad = S_{l,m}^{\nu_2}(I - \beta_m(h^t \otimes h^t)\{I - M_{l-1,2m}(\nu_1, \nu_2)\} A_{l-1,2m}^{-1}(h \otimes h) A_{l,m} \\ &\qquad - \beta_m(g^t \otimes h^t)\{I - M_{l-1,2m+1}(\nu_1, \nu_2)\} A_{l-1,2m+1}^{-1}(g \otimes h) A_{l,m}) S_{l,m}^{\nu_1} \end{aligned} \tag{3.7}$$

with $M_{L_e,m}(\nu_1, \nu_2) = 0$.

Proof: The recursion (3.7) follows by transferring the proof of Lemma 7.1.4 in [10] to our situation.

3.2 Robustness of the Damped V-Cycle

For our considerations we will need the Euclidean (spectral) norm $\|w\| = \langle w, w\rangle^{1/2}$ as well as the energy-norms relative to the $A_{l,m}$'s, $\|w\|_{l,m} = \langle w, w\rangle_{l,m}^{1/2} := \langle A_{l,m} w, w\rangle^{1/2}$. The associated matrix norms are denoted by the same symbols. With $\varrho(B)$ we abbreviate the spectral radius of B which is the largest absolute value of the eigenvalues of B.

The following proposition has been proved in [16].

Proposition 3.2. *There exist positive numbers $\sigma_m < 1$ being independent of ε and δ_l such that*

$$\varrho(A_{l-1,2m}^{-1} B_{l,m} A_{l-1,2m+1}^{-1} B_{l,m}^t) \leq \sigma_m \tag{3.8}$$

with $B_{l,m} = (h \otimes h) A_{l,m}(g^t \otimes h^t) = \varepsilon h c_l^m g^t \otimes I$. Further, the strong Cauchy inequality

$$|\langle w, v\rangle_{l,m}| \leq \sqrt{\sigma_m}\|w\|_{l,m}\|v\|_{l,m}$$

holds true for all w, $v \in \mathbf{R}^{n_l^2}$ satisfying $w = (h^t h \otimes h^t h) w$ and $v = (g^t g \otimes h^t h) v$, respectively.

The numbers σ_m will play a critical role in our convergence analysis of the V-cycle presented in this section. They are uniformly bounded smaller than 1, i.e.

$$\sigma := \sup\{\sigma_m | m \in \mathbf{N}_0\} < 1, \tag{3.9}$$

which will be proved in Lemma A.2 (Appendix). Table 1 shows the first a few σ_m's for the wavelet orders $N = 3$ and $N = 4$.

Table 1. The numbers σ_m for the Daubechies wavelets of order 3 and 4

	σ_0	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
$N = 3$	2.9E-1	2.8E-2	2.4E-3	2.3E-2	1.9E-4	1.6E-3	5.5E-3	5.0E-3
$N = 4$	1.9E-1	8.5E-3	6.9E-4	6.3E-3	5.0E-5	5.1E-4	1.3E-3	1.4E-4

From now on we require a symmetry condition for our BIM, that is, the matrix $W_{l,m}$ in (3.3) satisfies

$$W_{l,m}^t = W_{l,m} \geq A_{l,m} \quad \text{and} \quad \|W_{l,m}\| \leq C_W \delta_l^{-2}. \tag{3.10}$$

The notation $A \leq B$ signifies that A and B are symmetric matrices and that $B - A$ is positive semi-definite.

Remark. The (sufficiently) damped Jacobi iteration as well as the symmetric Gauß-Seidel iteration fulfill (3.10), see e.g. [10].

We introduce the abbreviations $p_0 = h \otimes h$, $p_1 = g \otimes h$ and the transformed matrices

$$\hat{M}_{l,m}(v_1, v_2) = A_{l,m}^{1/2} M_{l,m}(v_1, v_2) A_{l,m}^{-1/2}, \quad \hat{p}_i = A_{l-1,2m+i}^{-1/2} p_i A_{l,m}^{1/2}, \quad i = 0, 1,$$
$$\hat{S}_{l,m} = A_{l,m}^{1/2} S_{l,m} A_{l,m}^{-1/2} = I - X_{l,m} \quad \text{with} \quad X_{l,m} = A_{l,m}^{1/2} W_{l,m}^{-1} A_{l,m}^{1/2}.$$

The recursion (3.7) implies

$$\hat{M}_{l,m}(v_1, v_2) = \hat{S}_{l,m}^{v_2} \left(I - \beta_m \sum_{i=0}^{1} \hat{p}_i^t \{ I - \hat{M}_{l-1,2m+i}(v_1, v_2) \} \hat{p}_i \right) \hat{S}_{l,m}^{v_1}.$$

Before we formulate the robustness result for the damped V-cycle in Theorem 3.8 we first supply some preparatory lemmas and corollaries.

Lemma 3.3. *If* $\beta_m \in [0, (1 + \sqrt{\sigma_m})^{-1}]$ *then*

$$0 \leq Q_{l,m} := I - \beta_m (\hat{p}_0^t \hat{p}_0 + \hat{p}_1^t \hat{p}_1) \leq I. \tag{3.11}$$

Proof: The relation (3.11) is equivalent to $0 \leq \beta_m (\hat{p}_0^t \hat{p}_0 + \hat{p}_1^t \hat{p}_1) \leq I$ which is satisfied if $\beta_m \in [0, \varrho(\hat{p}_0^t \hat{p}_0 + \hat{p}_1^t \hat{p}_1)^{-1}]$ because $\hat{p}_0^t \hat{p}_0 + \hat{p}_1^t \hat{p}_1$ is positive semidefinite. We have that $\hat{p}_0^t \hat{p}_0 + \hat{p}_1^t \hat{p}_1 = A_{l,m}^{1/2} U^t D^{-1} U A_{l,m}^{1/2}$ with the 2×2-block diagonal matrix $D = \mathrm{diag}(A_{l-1,2m}, A_{l-1,2m+1})$ and with the mapping $U\colon \mathbf{R}^{n_l^2} \to \mathbf{R}^{n_l^2/2}$ defined by

$$U := \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} h \otimes h \\ g \otimes h \end{pmatrix}. \tag{3.12}$$

Now, let us estimate the spectral radius $\varrho(A_{l,m}^{1/2} U^t D^{-1} U A_{l,m}^{1/2})$:

$$\begin{aligned} \varrho(A_{l,m}^{1/2} U^t D^{-1} U A_{l,m}^{1/2}) &= \varrho(D^{-1} U A_{l,m} U^t) = \varrho(I - (I - D^{-1} U A_{l,m} U^t)) \\ &\leq 1 + \varrho(I - D^{-1} U A_{l,m} U^t). \end{aligned}$$

The estimate (3.8) in Proposition 3.2 implies that

$$r_m(\varepsilon, \delta_l) := \varrho(I - D^{-1} U A_{l,m} U^t) \leq \sqrt{\sigma_m}. \tag{3.13}$$

Lemma 3.4. *Suppose* (3.10) *and let* $\tilde{Q}_{l,m} := I - A_{l,m}^{1/2} U^t (U A_{l,m} U^t)^{-1} U A_{l,m}^{1/2}$. *Then,*

$$0 \leq \tilde{Q}_{l,m} \leq C X_{l,m}, \tag{3.14}$$

where C *is a constant being independent of* ε, δ_l *and* m.

Proof: We follow the proof of Lemma 6.4.6 in [10]. The statement of Lemma B.1 (Appendix) implies $\tilde{Q}_{l,m} \leq C_A \delta_l^2 A_{l,m}$ and (3.10) can be rewritten as $I \leq C_W \delta_l^{-2} W_{l,m}^{-1}$ which gives $A_{l,m} \leq C_W \delta_l^{-2} X_{l,m}$. Hence, (3.14) holds with $C = C_A C_W$.

Corollary 3.5. *Suppose* (3.10) *and let* $\beta_m \in [0, (1+\sqrt{\sigma_m})^{-1}]$. *Then,*

$$0 \le Q_{l,m} \le CX_{l,m} + d_m I, \tag{3.15}$$

where $d_m = 1 - \beta_m(1-\sqrt{\sigma_m})$ *and where* C *is the constant from* (3.14).

Proof: We have that $Q_{l,m} - \tilde{Q}_{l,m} = A_{l,m}^{1/2}U^t((UA_{l,m}U^t)^{-1} - \beta_m D^{-1})UA_{l,m}^{1/2}$ with U and D as in the proof of Lemma 3.3. Further,

$$\begin{aligned}\varrho(Q_{l,m} - \tilde{Q}_{l,m}) &= \varrho(I - \beta_m D^{-1}UA_{l,m}U^t)\\ &= \varrho((1-\beta_m)I + \beta_m(I - D^{-1}UA_{l,m}U^t))\\ &\le 1 - \beta_m + \beta_m\varrho(I - D^{-1}UA_{l,m}U^t)\\ &\le 1 - \beta_m + \beta_m\sqrt{\sigma_m} = d_m,\end{aligned}$$

where we used (3.13). From (3.14) follows that

$$\begin{aligned}0 \le Q_{l,m} &\le \tilde{Q}_{l,m} + \varrho(Q_{l,m} - \tilde{Q}_{l,m})I\\ &\le CX_{l,m} + d_m I.\end{aligned}$$

□

We set $\hat{M}_{l,m} := \hat{M}_{l,m}(v/2, v/2)$ which formally makes sense for any real $v \ge 0$.

Lemma 3.6. *Suppose* (3.10). *Let* $\beta_m \in [0,(1+\sqrt{\sigma_m})^{-1}]$ *and let* $d_m = 1 - \beta_m(1-\sqrt{\sigma_m})$. *If*

$$0 \le \hat{M}_{l-1,2m+i} \le \xi_{l-1,2m+i}I, \quad 0 \le \xi_{l-1,2m+i} \le 1, \quad i = 0, 1, \tag{3.16}$$

then

$$0 \le \hat{M}_{l,m} \le \xi_{l,m}I, \quad \xi_{l,m} = \min_{d_m(1-\bar{\xi}_{l,m})+\bar{\xi}_{l,m} \le y \le 1}\ \max_{0 \le x \le 1} f(x,y), \tag{3.17}$$

with $\bar{\xi}_{l,m} = \max\{\xi_{l-1,2m}, \xi_{l-1,2m+1}\}$ *and* $f(x,y) = (1-x)^v(y + (1-y)Cx/(1-d_m))$ *where* C *is the constant from* (3.14).

Proof: The proof is similar to the proof of Lemma 7.2.1 in [10].

Since $\hat{M}_{l,m} = \hat{S}_{l,m}^{v/2}Q_{l,m}\hat{S}_{l,m}^{v/2} + \beta_m\hat{S}_{l,m}^{v/2}\sum_{i=0}^{1}\hat{p}_i^t\hat{M}_{l-1,2m+i}\hat{p}_i\hat{S}_{l,m}^{v/2}$ and since $\hat{M}_{l-1,2m+i} \ge 0$, $i = 0, 1$, (3.11) proves $\hat{M}_{l,m} \ge 0$. Using (3.16) and (3.11) we obtain

$$\begin{aligned}\hat{M}_{l,m} &\le \hat{S}_{l,m}^{v/2}\{I - \beta_m\hat{p}_0^t\hat{p}_0(1-\xi_{l-1,2m}) - \beta_m\hat{p}_1^t\hat{p}_1(1-\xi_{l-1,2m+1})\}\hat{S}_{l,m}^{v/2}\\ &\le \hat{S}_{l,m}^{v/2}\{I - (1-\bar{\xi}_{l,m})\beta_m(\hat{p}_0^t\hat{p}_0 + \hat{p}_1^t\hat{p}_1)\}\hat{S}_{l,m}^{v/2}\\ &= \hat{S}_{l,m}^{v/2}\{\bar{\xi}_{l,m}I + (1-\bar{\xi}_{l,m})Q_{m,l}\}\hat{S}_{l,m}^{v/2}.\end{aligned}$$

By (3.15) and (3.11) we get $0 \le Q_{l,m} \le \alpha CX_{l,m} + (\alpha d_m + (1-\alpha))I$ for all $\alpha \in [0,1]$. Hence,

$$\begin{aligned}\hat{M}_{l,m} &\le \hat{S}_{l,m}^{v/2}\{\alpha(1-\bar{\xi}_{l,m})CX_{l,m} + (1 - \alpha(1-\bar{\xi}_{l,m})(1-d_m))I\}\hat{S}_{l,m}^{v/2}\\ &= \hat{S}_{l,m}^{v/2}\left(\frac{1-y}{1-d_m}CX_{l,m} + yI\right)\hat{S}_{l,m}^{v/2}\end{aligned}$$

for all $y \in [d_m(1-\bar{\xi}_{l,m}) + \bar{\xi}_{l,m}, 1]$. The statement (3.17) follows by $0 \le X_{l,m} \le I$.

Corollary 3.7. *We adopt the assumptions and notations of Lemma 3.6. Further, let* $\beta_m = (1 + \sqrt{\sigma_m})^{-1}$. *Then, there exists a positive integer* $\bar{v}$ *independent of* ε, δ_l *and* m *such that*

$$\varrho(\hat{M}_{l,m}) \leq \xi_{l,m} = 1 - \frac{1 - \sqrt{\sigma_m}}{1 + \sqrt{\sigma_m}} \min\{1 - \xi_{l-1,2m}, 1 - \xi_{l-1,2m+1}\}$$
$$= 1 - \frac{1 - \sqrt{\sigma_m}}{1 + \sqrt{\sigma_m}}(1 - \max\{\xi_{l-1,2m}, \xi_{l-1,2m+1}\})$$

for all $v \geq \bar{v}$.

Proof: The partial derivative $f_x(x, y) = (1 - x)^{v-1}(-(v + 1)(1 - y)Cx/(1 - d_m) - vy + (1 - y)C/(1 - d_m))$ shows that $f_x(x, y) \leq 0$ for $y \in [d_m(1 - \bar{\xi}_{l,m}) + \bar{\xi}_{l,m}, 1]$ if $v \geq \bar{v}$ with $\bar{v}$ sufficiently large. The lower bound $\bar{v}$ does not depend on m because the σ_m's are uniformly bounded smaller than 1, see (3.9). The function f is monotonically decreasing in x. Using the statement of Lemma 3.6 we estimate

$$\xi_{l,m} = \min_{d_m(1-\bar{\xi}_{l,m})+\bar{\xi}_{l,m} \leq y \leq 1} f(0, y) = \min_{d_m(1-\bar{\xi}_{l,m})+\bar{\xi}_{l,m} \leq y \leq 1} y$$
$$= 1 - \beta_m(1 - \sqrt{\sigma_m})(1 - \bar{\xi}_{l,m}). \qquad \square$$

Now, we are able to formulate and to prove our main result which shows that the convergence rate of the damped V-cycle of the FDMGM depends at most on the number of levels.

Theorem 3.8. *Let* $M_{L,0}(v, v)$ *be the iteration matrix of the iteration process* (3.5) *with damping factors* $\beta_m = (1 + \sqrt{\sigma_m})^{-1}$. *Further, let* $\bar{v}$ *be the positive integer determined in Corollary* 3.7. *If* $2v \geq \bar{v}$ *then*

$$\varrho(M_{L,0}(v, v)) = \|M_{L,0}(v, v)\|_{L,0} \leq 1 - \left(\frac{1 - \sqrt{\sigma}}{1 + \sqrt{\sigma}}\right)^{L-L_e},$$

where $0 \leq \sigma < 1$ *is defined in* (3.9).

Proof: We have that $\varrho(M_{l,m}(v, v)) = \varrho(\hat{M}_{l,m}(v, v)) = \|M_{l,m}(v, v)\|_{l,m}$. Since $(1 - \sqrt{\sigma_m})/(1 + \sqrt{\sigma_m}) \geq (1 - \sqrt{\sigma})/(1 + \sqrt{\sigma})$ for all m and since $\xi_{L_e,m} = 0$, $0 \leq m \leq 2^{L-L_e} - 1$, an inductive application of Corollary 3.7 proves Theorem 3.8.

Remark: The statement of Theorem 3.8 holds also true if we damp uniformly, i.e. $\beta_m \equiv (1 + \sqrt{\sigma})^{-1}$.

We have numerical evidence that $\sigma = \sigma_0$ holds true, e.g. see Table 1.

In Fig. 2 the convergence rates of the FDMGM (3.5) are plotted for $L \in \{3, 4, 5\}$ with $L_e = 0$ and for the uniform damping parameter $\beta = \beta_m = 0.9$. The underlying Daubechies wavelet packets have order $N = 3$. Again, the damped Jacobi iteration was used as BIM with the damping parameters described in Section 3.1 and with $v_1 = 2$, $v_2 = 0$. The choice $v_1 = v_2 = 1$ leads to the same convergencer rates. If we replace the Jacobi iteration by the symmetric Gauß-Seidel iteration then we have better convegence rates. However, the dependence on the levels remains unchanged.

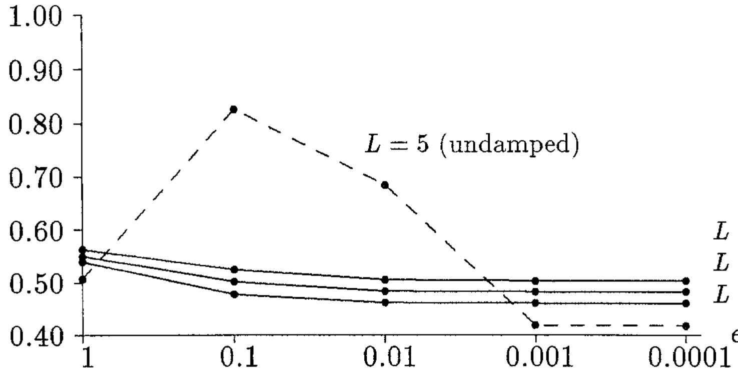

Figure 2. Damped V-cycle convergence rates of the wavelet variation of the FDMGM for different discretization step-sizes and for a uniform damping parameter $\beta = 0.9$

In principle, one can compute reasonable approximations to the β_m's by explicit formulas for the σ_m's, see [16], in an efficient way. This parameter choice gives worse convergence rates than the presented rates for the uniform damping parameter $\beta = 0.9$ which is larger than the maximum $\beta_0 = 0.65$ of the β_m's for $N = 3$. One reason for this behaviour we find in (3.13) where we see that an optimal damping parameter should depend on m, ε and δ_l. For instance, if δ_l is fixed then $r_m(\varepsilon, \delta_l)$ tends to zero as $\varepsilon \to 0$. Further, an explicit expression of r_m given in [16] shows that the assumption $r_m(\varepsilon, \delta_l) = 0$ is reasonable if $\varepsilon \ll \delta_l^2$. Moreover, in the extreme case $\varepsilon = 0$ where no damping is needed the proofs in Chapter 7.2 of [10] can be transfered to the FDMGM without any modifications and yield the contraction number

$$\|M_{L,0}(\nu_1, \nu_2)\|_{L,0} \le \frac{C}{\sqrt{C + \nu_1}\sqrt{C + \nu_2}}.$$

4. Conclusion

In this paper, we showed that the convergence rate of the sufficiently damped V-cycle of the wavelet variation of the FDMGM depends at most on the number of levels (Theorem 3.8). This is a first V-cycle convergence result for the FDMGM introduced in [11] and [12]. The presented result might be improved by chosing another damping strategy than (3.4), that is, both branches of the coarse grid correction are damped separately, and by investigating the dependence of the damping parameters on m, ε and δ_l more carefully.

Appendix

A. Uniform Boundedness of the σ_m's

In this Appendix the proof of (3.9) will be given.

We define

$$H(\omega) = \frac{1}{2} \cdot \sum_{k=0}^{2N-1} a_k e^{-ik\omega}, \quad i = \sqrt{-1}, \tag{A.1}$$

with the coefficients a_k from (2.1). The trigonometric polynomial H satisfies

$$H^2(\omega) + H^2(\omega + \pi) = 1, \tag{A.2}$$

see [6] or [7]. Further we will need the positive cosine series

$$\lambda^m(\omega) = \Gamma_0^m + 2 \sum_{k=1}^{2N-2} \Gamma_k^m \cos(k\omega) \tag{A.3}$$

with the second order connection coefficients Γ_k^m (2.4). We have the following recursion formula

$$\lambda^{2m}(\omega) = 4(|H(\omega/2)|^2 \lambda^m(\omega/2) + |H(\omega/2 + \pi)|^2 \lambda^m(\omega/2 + \pi)), \tag{A.4}$$

$$\lambda^{2m+1}(\omega) = 4(|H(\omega/2 + \pi)|^2 \lambda^m(\omega/2) + |H(\omega/2)|^2 \lambda^m(\omega/2 + \pi)), \tag{A.5}$$

see [16]. With these definitions we are able to express the numbers σ_m from Proposition 3.2 by

$$\sigma_m = \sup_{0 \leq \omega \leq 2\pi} \frac{R^m(\omega)}{R^m(\omega) + \lambda^m(\omega/2)\lambda^m(\omega/2 + \pi)} \tag{A.6}$$

where $R^m(\omega) = |H(\omega/2)|^2 |H(\omega/2 + \pi)|^2 (\lambda^m(\omega/2) - \lambda^m(\omega/2 + \pi))^2$, see also [16].

In the lemma below the bracket expression $[\cdot]$ denotes the 'greatest integer' and ld denotes the logarithm with respect to the basis 2.

Lemma A.1. *The inclusion*

$$0 < \min_{0 \leq \omega \leq 2\pi} \lambda^1(\omega) \leq 4^{-[\mathrm{ld}\, m]} \lambda^m(\omega) \leq \max_{0 \leq \omega \leq 2\pi} \lambda^1(\omega) < \infty \tag{A.7}$$

holds true for all $m \geq 1$.

Proof: Both inner inequalities follow inductively from (A.4), (A.5) and (A.2). The leftmost inequality was shown in Lemma 2.3 (iv) in [16].

Lemma A.2. *The supremum of the* σ_m*'s* (A.6) *is bounded smaller than* 1,

$$\sup\{\sigma_m | m \in \mathbf{N}_0\} < 1.$$

Proof: We assume that $\sup\{\sigma_m | m \geq 1\} = 1$. Then, for any $\alpha > 0$ there exists an integer $m_\alpha \geq 1$ and an $\omega_\alpha \in [0, 2\pi)$ such that

$$\sigma_{m_\alpha} = \frac{R^{m_\alpha}(\omega_\alpha)}{R^{m_\alpha}(\omega_\alpha) + \lambda^{m_\alpha}(\omega_\alpha/2)\lambda^{m_\alpha}(\omega_\alpha/2 + \pi)} \geq 1 - \alpha,$$

which is equivalent to

$$\alpha \geq \frac{16^{-[\mathrm{ld}\, m_\alpha]} \lambda^{m_\alpha}(\omega_\alpha/2)\lambda^{m_\alpha}(\omega_\alpha/2 + \pi)}{16^{-[\mathrm{ld}\, m_\alpha]} R^{m_\alpha}(\omega_\alpha) + 16^{-[\mathrm{ld}\, m_\alpha]} \lambda^{m_\alpha}(\omega_\alpha/2)\lambda^{m_\alpha}(\omega_\alpha/2 + \pi)} > 0. \tag{A.8}$$

By (A.7) and by $|H(\omega/2)|^2|H(\omega/2+\pi)|^2 < 1/4$ which follows from (A.2) we derive $0 < 16^{-[\mathrm{ld}\, m_\alpha]} R^{m_\alpha}(\omega_\alpha) \le (\max_{0\le\omega\le 2\pi} \lambda^1(\omega))^2$. Therefore, (A.8) implies that $16^{-[\mathrm{ld}\, m_\alpha]}\lambda^{m_\alpha}(\omega_\alpha/2)\lambda^{m_\alpha}(\omega_\alpha/2+\pi) \to 0$ as α tends to zero. However, this contradicts $4^{-[\mathrm{ld}\, m]}\lambda^m(\omega) \ge \min_{0\le\omega\le 2\pi}\lambda^1(\omega) > 0$, see (A.7). Hence, $\sup\{\sigma_m | m \ge 1\} < 1$. Since $\sigma_0 < 1$ Lemma A.2 is proved.

B. An Approximation Property

Our goal is to verify the approximation property stated in Lemma B.1.

Lemma B.1. *Let the matrices $A_{l,m}$ be defined as in* (3.1) *and the mapping U as in* (3.12). *Then,*

$$\|A_{l,m}^{-1} - U^t(UA_{l,m}U^t)^{-1}U\| \le C_A\delta_l^{-2},$$

where C_A is a constant being independent of ε, ε_l and m.

First we note that $\lambda_\mu^m = \delta_l^{-2}\lambda^m(2\pi/\mu/n_l)$, $\mu = 0, \ldots, n_l - 1$, are the eigenvalues of c_l^m (3.2) where the function λ^m is given in (A.3). All circulant matrices of the dimension n_l share the same system of l^2-orthonormal eigenvectors v_μ, $\mu = 0, \ldots, n_l - 1$,

$$(v_\mu)_k = \frac{1}{\sqrt{n_l}} e^{-i2\pi\mu k/n_l},$$

see [8]. Consequently, the matrices $A_{l,m}$ have the eigenvalues

$$\Lambda_{m;\mu,\nu} = \delta_l^{-2}(\varepsilon\lambda^m(2\pi\mu/n_l) + \lambda^0(2\pi\nu/n_l)) + 1,$$

$0 \le \mu, \nu \le n_l - 1$, with corresponding eigenvectors $v_{\mu,\nu} = v_\mu \otimes v_\nu$.

Lemma B.2. *We have that*

$$\|U^t(UA_{l,m}U^t)^{-1}UA_{l,m}\| \le C,$$

where the constant C does not depend on ε, δ_l or m.

Proof: The proof will be very technical and we will use results proved in [16].

The matrix $UA_{l,m}U^t$ has the following block structure

$$UA_{l,m}U^t = \begin{pmatrix} A_{l-1,2m} & B \\ B^t & A_{l-1,2m+1} \end{pmatrix}$$

with $B = \varepsilon c_{l-1}^{m;c} \otimes I$ and $c_{l-1}^{m;c} = hc_l^m g^t$. Again, $c_{l-1}^{m;c}$ is a circulant matrix,

$$c_{l-1}^{m,c} = \delta_{l-1}^{-2}\,\mathrm{Cir}_{n_{l-1}}(\Gamma_0^{m,c} \quad \Gamma_1^{m,c} \quad \cdots \quad \Gamma_{2N-2}^{m,c} \quad 0 \quad \cdots \quad 0 \quad \Gamma_{2-2N}^{m,c} \quad \cdots \quad \Gamma_{-1}^{m,c}),$$

determined by the (mixed) connection coefficients

$$\Gamma_k^{m,c} = \int_{\mathbf{R}} (\psi^{2m})'(x-k)(\psi^{2m+1})'(x)\,dx.$$

The eigenvalues of $c_{l-1}^{m;c}$ are denoted by $\lambda_\mu^{m,c} = \delta_{l-1}^{-2}\lambda^{m,c}(2\pi\mu/n_{l-1}), 0 \le \mu \le n_{l-1} - 1$, where $\lambda^{m,c}(\omega) = \sum_{k=2-2N}^{2N-2}\Gamma_k^{m,c}e^{-ik\omega}$ satisfying

$$\lambda^{m,c}(\omega) = H(\omega/2)\overline{H(\omega/2+\pi)}(\lambda^m(\omega/2) - \lambda^m(\omega/2+\pi)), \tag{B.1}$$

with H as in (A.1). The inverse of $UA_{l,m}U^t$ can be expressed by its block entries

$$(UA_{l,m}U^t)^{-1} = \begin{pmatrix} A_0^{-1}(I - BA_1^{-1}\Sigma^{-1}B^tA_0^{-1}) & -A_0^{-1}BA_1^{-1}\Sigma^{-1} \\ -A_1^{-1}\Sigma^{-1}B^tA_0^{-1} & A_1^{-1}\Sigma^{-1} \end{pmatrix} \tag{B.2}$$

where $\Sigma = I - A_1^{-1}B^tA_0^{-1}B$. For convenience we set $A_0 := A_{l-1,2m}$ and $A_1 := A_{l-1,2m+1}$. The matrix Σ is invertible because

$$\varrho(I - \Sigma) \le \sigma < 1 \tag{B.3}$$

due to (3.8) and (3.9). Let f be in $\mathbf{R}^{n_l^2}$. Then,

$$\begin{aligned} \|U^t(UA_{l,m}U^t)^{-1}UA_{l,m}f\|^2 = \sum_{\mu,\nu}\sum_{\alpha,\beta} & \Lambda_{m;\mu,\nu}\Lambda_{m;\alpha,\beta} \\ & \times \langle (UA_{l,m}U^t)^{-1}Uv_{\mu,\nu}, (UA_{l,m}U^t)^{-1}U\bar{v}_{\alpha,\beta}\rangle \langle f, v_{\mu,\nu}\rangle \\ & \times \langle f, \bar{v}_{\alpha,\beta}\rangle. \end{aligned}$$

A simple calculation shows that

$$Uv_{\mu,\nu} = \begin{pmatrix} H(\omega_\mu)H(\omega_\nu)\tilde{v}_{2\mu,2\nu} \\ G(\omega_\mu)H(\omega_\nu)\tilde{v}_{2\mu,2\nu} \end{pmatrix}. \tag{B.4}$$

Here, $\omega_\mu = 2\pi\mu/n_l$ and $G(\omega) = 2^{-1}\sum_{k=0}^{2N-1} b_k e^{-ik\omega}$ with the b_k's from (2.3). The vectors $\tilde{v}_{2\mu,2\nu}$ are tensor products, $\tilde{v}_{2\mu,2\nu} = \tilde{v}_{2\mu} \otimes \tilde{v}_{2\nu}$, where $(\tilde{v}_{2\mu})_k = \sqrt{2}\,e^{-i2\pi 2\mu k/n_l}/\sqrt{n_l}$, $k = 0, \ldots, n_{l-1} - 1$. Note that $\tilde{v}_{2\mu,2\nu}$ is an eigenvector of the circulant matrices A_0, A_1, B and B^t, respectively. Using (B.4) as well as (B.2) leads to

$$(UA_{l,m}U^t)^{-1}Uv_{\mu,\nu} = \begin{pmatrix} r_{\mu,\nu}\tilde{v}_{2\mu,2\nu} \\ t_{\mu,\nu}\tilde{v}_{2\mu,2\nu} \end{pmatrix}$$

with

$$\begin{aligned} r_{\mu,\nu} &= H(\omega_\mu)H(\omega_\nu)\Lambda^{-1}_{2m;2\mu,2\nu}(1 + \Lambda^{-1}_{\Sigma;2\mu,2\nu} \\ &\quad \times (\Lambda^{-1}_{2m+1;2\mu,2\nu}|\Lambda^c_{2m;2\mu,2\nu}|^2\Lambda^{-1}_{2m;2\mu,2\nu} - \Lambda^c_{2m;2\mu,2\nu}\Lambda^{-1}_{2m+1;2\mu,2\nu})), \\ t_{\mu,\nu} &= G(\omega_\mu)H(\omega_\nu)\Lambda^{-1}_{2m+1;2\mu,2\nu}\Lambda^{-1}_{\Sigma;2\mu,2\nu}(-\bar{\Lambda}^c_{2m;2\mu,2\nu}\Lambda^{-1}_{2m;2\mu,2\nu} + 1), \end{aligned}$$

where $\Lambda^c_{2m;2\mu,2\nu} = \varepsilon\delta_{l-1}^{-2}\lambda^{m,c}(2\omega_\mu)$ and $\Lambda_{\Sigma;2\mu,2\nu} = 1 - \Lambda^{-1}_{2m;2\mu,2\nu}|\Lambda^c_{2m;2\mu,2\nu}|^2\Lambda^{-1}_{2m+1;2\mu,2\nu}$. Since

$$\begin{aligned} \langle (UA_{l,m}U^t)^{-1}Uv_{\mu,\nu}, (UA_{l,m}U^t)^{-1}U\bar{v}_{\alpha,\beta}\rangle &= (r_{\mu,\nu}\bar{r}_{\alpha,\beta} + t_{\mu,\nu}\bar{t}_{\alpha,\beta})\langle \tilde{v}_{2\mu,2\nu}, \bar{\tilde{v}}_{2\alpha,2\beta}\rangle \\ &= (r_{\mu,\nu}\bar{r}_{\alpha,\beta} + t_{\mu,\nu}\bar{t}_{\alpha,\beta})\Delta^{\mu,\alpha}_{n_l}\Delta^{\nu,\beta}_{n_l} \end{aligned}$$

(here $\Delta^{\mu,\alpha}_{n_l} = 1$ if $|\mu - \alpha| = kn_l/2$, $k \in \mathbf{Z}$, ad $\Delta^{\mu,\alpha}_{n_l} = 0$ otherwise) we have that

$$\|U^t(UA_{l,m}U^t)^{-1}UA_{l,m}f\| \le 3\|f\| \max_{\mu,\nu} \Lambda_{m;\mu,\nu}(|r_{\mu,\nu}| + |t_{\mu,\nu}|).$$

In the last step of the proof we show that the maximum is independent of ε, δ_l and m. Therefore we supply a bunch of estimates. First,

$$\Lambda_{\Sigma;2\mu,2\nu} \ge 1 - \sigma \quad \text{and} \quad \Lambda^{-1}_{2m;2\mu,2\nu}|\Lambda^c_{2m;2\mu,2\nu}|^2\Lambda^{-1}_{2m+1;2\mu,2\nu} \le \sigma \tag{B.5}$$

by (B.3) and (3.8), (3.9), respectively. Further,

$$\frac{|H(\omega_\mu)H(\omega_\nu)|\Lambda_{m;\mu,\nu}}{\Lambda_{2m;2\mu,2\nu}} \le 4 \sup_{0\le\omega,\eta\le 2\pi} \frac{|H(\omega)H(\eta)|(\varepsilon\lambda^m(\omega)+\lambda^0(\eta)+\delta_l^2)}{\varepsilon\lambda^{2m}(2\omega)+\lambda^0(2\eta)+\delta_{l-1}^2}$$
$$\le 4\max\left\{\underbrace{\sup_{0\le\omega\le 2\pi}\frac{|H(\omega)|\lambda^m(\omega)}{\lambda^{2m}(2\omega)}}_{=E_1}, \underbrace{\sup_{0\le\eta\le 2\pi}\frac{|H(\eta)|\lambda^0(\eta)}{\lambda^0(2\eta)}}_{=E_2}, \frac{\delta_l^2}{\delta_{l-1}^2}\right\}.$$

If $m \ge 1$ then (A.7) implies that E_1 is uniformly bounded in m. For $m = 0$, E_1 equals E_2. Due to part (ii) of Lemma 2.3 in [16] $\lambda^0(2\omega^*)$ is equal to zero if and only if $\omega^* \in \{0, \pi, 2\pi\}$. For $\omega^* \in \{0, 2\pi\}$, the limit $\lim_{\omega\to\omega^*}\lambda^0(\omega)/\lambda^0(2\omega)$ exists. In the case $\omega^* = \pi$, the function $H(\omega)$ has a zero at least of order 3 ($N \ge 3$), see [6] or [7]. However, the zero of $\lambda^0(2\omega)$ in $\omega^* = \pi$ is only of order 2, Lemma 2.1 (i) in [16]. Consequently, E_1 and E_2 are finite and hence,

$$\frac{|H(\omega_\mu)H(\omega_\nu)|\Lambda_{m;\mu,\nu}}{\Lambda_{2m;2\mu,2\nu}} \le C_1, \quad C_1 \ne C_1(\varepsilon,\delta_l,m). \tag{B.6}$$

Next,

$$\frac{|G(\omega_\mu)H(\omega_\nu)|\Lambda_{m;\mu,\nu}}{\Lambda_{2m+1;2\mu,2\nu}} \le 4 \sup_{0\le\omega,\eta\le 2\pi} \frac{|G(\omega)H(\eta)|(\varepsilon\lambda^m(\omega)+\lambda^0(\eta)+\delta_l^2)}{\varepsilon\lambda^{2m+1}(2\omega)+\lambda^0(2\eta)+\delta_{l-1}^2}$$
$$\le 4\max\left\{\underbrace{\sup_{0\le\omega\le 2\pi}\frac{\lambda^m(\omega)}{\lambda^{2m+1}(2\omega)}}_{=E_3}, \underbrace{\sup_{0\le\eta\le 2\pi}\frac{|H(\eta)|\lambda^0(\eta)}{\lambda^0(2\eta)}}_{=E_2}, \frac{\delta_l^2}{\delta_{l-1}^2}\right\}.$$

Using (A.7) we are able to estimate $E_3 \le \max_\omega \lambda^1(\omega)/\min_\omega \lambda^1(\omega)$ for $m \ge 1$. Hence, E_3 is uniformly bounded in m and

$$\frac{|G(\omega_\mu)H(\omega_\nu)|\Lambda_{m;\mu,\nu}}{\Lambda_{2m+1;2\mu,2\nu}} \le C_2, \quad C_2 \ne C_2(\varepsilon,\delta_l,m). \tag{B.7}$$

Finally, let j be 0 or 1, then

$$\frac{|\Lambda^c_{2m;2\mu,2\nu}|}{\Lambda_{2m+j;2\mu,2\nu}} \le 4 \sup_{0\le\omega,\eta\le 2\pi} \frac{\varepsilon|\lambda^{m,c}(2\omega)|}{\varepsilon\lambda^{2m+j}(2\omega)+\lambda^0(2\eta)+\delta_{l-1}^2}$$
$$\le 4 \underbrace{\sup_{0\le\omega\le 2\pi}\frac{|\lambda^{m,c}(2\omega)|}{\lambda^{2m+j}(2\omega)}}_{=E_4}.$$

We have that $|\lambda^{m,c}(2\omega)| \le \max_\omega \lambda^m(\omega)$ by (B.1) and by $|H(\omega)H(\omega+\pi)| \le 1/2$. The cases $j = 1$ and $j = 0$ with $m \ge 1$ yield $E_4 \le \max_\omega \lambda^1(\omega)/\min_\omega \lambda^1(\omega)$. It remains to consider $j = 0$ and $m = 0$. However, this was already done in Lemma 2.4 of [16]. Altogether we have shown that

$$\frac{|\Lambda^c_{2m;2\mu,2\nu}|}{\Lambda_{2m+j;2\mu,2\nu}} \le C_3, \quad C_3 \ne C_3(\varepsilon,\delta_l,m), \tag{B.8}$$

for $j = 0, 1$. We finish the proof of Lemma B.2 by

$$\Lambda_{m;\mu,\nu}|r_{\mu,\nu}| \leq C_1(1 + (1 - \sigma)^{-1}(\sigma + C_3)),$$

$$\Lambda_{m;\mu,\nu}|t_{\mu,\nu}| \leq C_2(1 - \sigma)^{-1}(C_3 + 1),$$

where we have used (B.5), (B.6), (B.7) and (B.8).

Proof of Lemma B.1: We will need the estimate $\|(I - U^t U)A_{l,m}^{-1}\| \leq C'\delta_l^2$ which can be verified by a simple modification of the proof of Lemma 5.3 in [16]. The constant C' does not depend on ε, δ_l or m. Now,

$$\begin{aligned}\|A_{l,m}^{-1} - U^t(UA_{l,m}U^t)^{-1}U\| &= \|(I - U^t(UA_{l,m}U^t)^{-1}UA_{l,m})(I - U^tU)A_{l,m}^{-1}\| \\ &\leq \|(I - U^t(UA_{l,m}U^t)^{-1}UA_{l,m})\|\,\|(I - U^tU)A_{l,m}^{-1}\| \\ &\leq C_A\delta_l^2\end{aligned}$$

with $C_A = (1 + C)C'$ where the constant C is as in Lemma B.2.

References

[1] Bank, R. E., Dupont, T. F., Yserentant, H.: The hierarchical basis multigrid method. Numer. Math. *52*, 427–458 (1988).
[2] Beylkin, G.: On the representation of operators in bases of compactly supported wavelets. SIAM J. Numer. Anal. *6*, 1716–1740 (1992).
[3] Bramble, J. H., Pasciak, J. E., Wang, J., Xu, J.: Convergence estimate for multigrid algorithms without regularity assumptions. Math. Comp., *57*, 427–458 (1991).
[4] Ciarlet, P. G.: The finite element methods for elliptic problems. New York: North-Holland 1987.
[5] Coifman, R. R., Meyer, Y., Wickerhauser, M. V.: Size properties of wavelet packets. In: Ruskai (ed.) Wavelets and their applications, pp. 453–470. Boston: Jones and Bartlett 1992.
[6] Daubechies, I.: Orthonormal bases of compactly supporte wavelets. Comm. Pure Appl. Math. *41*, 906–966 (1988).
[7] Daubechies, I.: Ten lectures on wavelets. CBMS-NSF Series in Applied Mathematics. Philadelphia: SIAM Publications 1992.
[8] Davis, P. J.: Circulant matrices. New York: John Wiley 1979.
[9] Eirola, T.: Sobolev characterization of solutions of dilation equations. SIAM J. Math. Anal. *23*, 1015–1030 (1992).
[10] Hackbusch, W.: Multi-grid methods and applications. Springer Series in Computational Mathematics. New York: Springer 1985.
[11] Hackbusch, W.: The frequency decomposition multi-grid method, part I: Application to anisotropic equations. Numer. Math. *56*, 229–245 (1989).
[12] Hackbusch, W.: The frequency decomposition multi-grid method, part II: Convergence analysis based on the additive Schwarz method. Numer. Math. *63*, 433–453 (1992).
[13] Latto, A., Resnikoff, H. L., Tenenbaum, E.: The evaluation of connection coefficients of compactly supported wavelets. In: Proceedings of the USA-French Workshop on Wavelets and Turbulence. Princeton University 1991.
[14] Mallat, S.: Multiresolution approximation and wavelet orthonormal bases of $L^2(\mathbf{R})$. Trans. Amer. Math. Soc. *315*, 69–87 (1989).
[15] Meyer, Y.: Wavelets: algorithms and applications. Philadelphia: SIAM Publications 1993.
[16] Rieder, A., Wells, R. O., Jr., Zhou, X.: A wavelet approach to robust multilevel solvers for anisotropic elliptic problems. Technical Report 93-07, Rice University, 1993. Computational Mathematics Laboratory. Accepted for publication in Appl. Comput. Harmonic Anal. (ACHA)
[17] Wang, J.: Convergence analysis without regularity assumptions for multigrid algorithms based on SOR smoothing. SIAM J. Numer. Anal. *29*, 987–1001 (1992).

[18] Wells, R. O., Jr., Zhou, X.: Wavelet interpolation and approximate solutions of elliptic partial differential equations. In: Wilson, R., Tanner, E. A. (eds.) Noncompact lie groups. Dordrecht: Kluwer, 1994. To appear. Proceedings of NATO Advanced Research Workshop.

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